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JHEP07(2009)041

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NLO antenna subtraction with massive fermions

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ABSTRACT: We present an extension of the antenna subtraction formalism at NLO to include massive final state fermions. The basic ingredients to the subtraction terms, the NLO massive final-final antenna functions are derived and integrated over the corresponding factorised phase space. Those antenna functions account for all soft, collinear and quasi-collinear limits of the QCD matrix elements involving massive fermions in the final state.

KEYWORDS: Jets, NLO Computations, QCD

ARXIV EPRINT: [0904.3297](https://arxiv.org/abs/0904.3297)

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1 Introduction

Hard scattering processes leading to final states with heavy particles are important observables for present and future colliders. Reliable theoretical predictions for these observables require the calculation of at least the next-to-leading order QCD corrections.

For hard scattering processes, involving massless or massive QCD partons, the perturbative corrections to a given process and at a given order in QCD are obtained when all partonic channels contributing to that order are summed. In general, each partonic channel contains both ultraviolet and infrared (soft and collinear) divergences. The ultraviolet poles are removed by renormalisation in each channel. The remaining soft and collinear poles cancel among each other when all partonic contributions are summed over [1]. At NLO, one has to combine virtual-one loop calculations with real emission contributions from unresolved partons. While infrared singularities from purely virtual corrections are obtained immediately after integration over the loop momenta, their extraction is more involved for the real emission contributions. Here, the infrared singularities only become explicit after integrating the real radiation matrix elements over the phase space. To build a general-purpose Monte Carlo program to evaluate observables at NLO requires therefore an analytic cancellation of infrared singularities before any numerical integration can be performed.

For the task of NLO calculations, several systematic and process-independent procedures are available and have been applied to a variety of processes [2]. The two main methods are phase space slicing [3, 4] and the subtraction methods [5, 6]. Except for the phase space slicing technique, all subtraction methods consist in introducing terms which are subtracted from the real radiation part at each phase space point. These subtraction terms approximate the matrix element in all singular limits and should be sufficiently simple to be integrated over the corresponding phase space part analytically. After this integration, the infrared divergences of the subtraction terms become explicit and the integrated subtraction terms can be added to the virtual corrections, thus yielding an infrared finite result.

One of the widely used subtraction formalisms at NLO is the dipole subtraction formalism of Catani and Seymour, which in its original formulation [6] deals with massless partons in final and/or initial state at NLO.

Another subtraction scheme is the antenna formalism [7–9], which constructs the subtraction terms from so-called antenna functions. The antenna functions describe all unresolved partonic (soft and collinear) radiation between a hard pair of colour-ordered partons, the radiators. These functions can be derived systematically from physical matrix elements, as shown in [10, 11] and can be integrated over the factorised phase space. In the antenna subtraction method, the subtraction terms are constructed from products of antenna functions with reduced matrix elements (with fewer partons than the original matrix element) such that these subtraction terms are acting on individual colour-ordered real radiation matrix elements. So far, this formalism can handle massless partons in final and/or initial state at NLO [7, 9] and it can treat massless final state parton radiation at NNLO [8]. It is the only formalism where the NNLO formulation has been worked out in full and applied to evaluate NNLO corrections to jet observables, namely for $e^+e^- \rightarrow 3$ jets [12, 13].

To evaluate hard scattering processes involving *massive* hard partons in the final state at the next-to-leading order level, a subtraction formalism is needed as well in order to isolate and cancel infrared divergences among different partonic contributions to the cross section. However the extension from massless partons to massive partons involve complications and new features. The kinematics is more involved due to the finite value of the parton masses.

Furthermore, QCD radiation from massive partons, although not leading to strict collinear divergences, (which are regulated by the mass of the massive particle), will be proportional to $\ln Q^2/M^2$, where M is the parton mass and Q is the typical scale of the hard scattering process. In kinematical configurations where $Q \gg M$, these logarithmically enhanced contributions become large and can spoil the numerical convergence of the calculation. Although these logarithmic terms cancel in the final NLO result they appear at intermediate steps of the calculation. Those can be easily traced though. Indeed, these logarithmic enhanced terms are related to a process-independent behaviour of the matrix elements; its singular behaviour in the massless limit ($M \rightarrow 0$). This singular behaviour is related to the *quasi-collinear* [14] limit of the matrix element. For massless partons, infrared divergences arise in soft and collinear kinematical configurations and in these configurations matrix element and phase space obey QCD factorisation formulae [15]. Similarly, for massive partons, matrix element and phase space will obey factorisation formulae in the soft and quasi-collinear limit.

The dipole subtraction formalism of Catani and Seymour has been extended to include effects of massive partons in [14, 16, 17] at NLO. An extension of the antenna subtraction method including the radiation effects of massive partons has been so far missing.

It is the purpose of this paper to present an extension of the antenna subtraction method to include radiation of final state massive fermions in order to be able to evaluate the production of massive particles beyond leading order within this formalism. For the sake of clarity, in this paper we restrict ourselves to the kinematical situation of a colour-neutral particle decaying into one or two coloured massive fermions of equal masses. In this situation, infrared singularities appear only due to final state radiation and only two scales are involved in the problem. As the radiators are in the final state only, the corresponding antenna functions involving those are regarded as final-final antennae.

The basic structure of the extension of the antenna subtraction formalism to include massive particles can be carried over to partons in the initial state. However, to deal with the production of massive final states at hadron colliders, the construction of the appropriate subtraction terms and their integrations cannot be solely performed with the ingredients presented in this paper. Besides massless initial-initial antenna functions derived in [9], which involve massless initial state radiators, and massive final-final antenna functions which will be derived in this paper, antenna functions involving a massless initial state and a massive final state are also required. As in the massless case, those initial-final antenna functions can be obtained from the final-final massive antenna functions by crossing one massless parton from the final to the initial state. Their integration over the corresponding phase space cannot be taken over from the final-final case, though.

Furthermore, the extension of the antenna subtraction method involving massive final state particles is not restricted to Standard Model processes. It can also be applied to the production of supersymmetric particles, for example. An appropriate treatment of massive final state scalars, like squarks could also be realised within this framework.

So far, final-final massless antennae have found some interesting applications: Starting from the three-parton antenna functions given in [8], a parton shower event generator VINCIA [18] has been constructed. One could therefore envisage a further implementation in VINCIA of the massive NLO final-final antennae presented in this paper. A combination between NLO calculation in the antenna subtraction formalism and parton showers could then become feasible for processes involving massless and massive final state fermions. Originally, parton showers based on the antennae approach for final state particles had been derived for the event generator ARIADNE [19]. Furthermore, parton showers related to the dipole subtraction formalism involving dipole functions were considered in the literature [20–22] and are already implemented [23] in SHERPA [24].

The paper will be organised as follows. In section 2, we recall the structure of the antenna subtraction terms at NLO for massless final state partons and state how it extends with the presence of massive partons. Section 3 presents the formulae for the massive antenna functions while in section 4 a list of all non-vanishing limits for these antennae is given. In section 5 we present the results for the integrated massive antenna functions. To validate our results for the integrated massive quark-antiquark antenna function, we recompute the NLO correction to the hadronic decay rate in section 6. Section 7 contains our conclusions.

2 Antenna subtraction with massive final-final configurations

In configurations involving final-final antennae, both radiators are in the final state. For massless radiators, this case was described in detail at NLO in [7] and NNLO in [8]. For massive radiators, the same factorisation formulae for the subtraction terms will hold. Only the ingredients, phase space and antenna functions will be modified to take into account the mass of radiators. For clarity reasons, we will keep the same notation as in [8].

The subtraction term for an unresolved parton j , emitted between massless or massive hard final-state radiators i and k reads,

$$d\sigma_{\text{NLO}}^S = \mathcal{N} \sum_{m+1} d\Phi_{m+1}(k_1, \dots, k_{m+1}; q) \frac{1}{S_{m+1}} \quad (2.1)$$

$$\sum_j X_{ijk}^0 |\mathcal{M}_m(k_1, \dots, K_I, K_K, \dots, k_{m+1}; q)|^2 J_m^{(m)}(k_1, \dots, K_I, K_K, \dots, k_{m+1}; q).$$

This subtraction term involves the m parton amplitude \mathcal{M}_m which depends only on the redefined on-shell momenta $k_1, \dots, k_I, k_K, \dots, k_{m+1}$. The momenta k_I, k_K are linear combinations of the momenta k_i, k_j, k_k while the tree-level antenna function X_{ijk}^0 depends only on k_i, k_j, k_k , X stands for different antenna types denoted (in the massive NLO case) by A, E, D or G depending on which radiators are involved. The jet function $J_m^{(m)}$ in eq. (2.2) does not depend on the individual momenta k_i, k_j and k_k , but only on K_I, K_K . This function ensures that any partonic contribution give rise to an m -jet final state.

The phase space $d\Phi_{m+1}$ can be factorised as follows,

$$d\Phi_{m+1}(k_1, \dots, k_{m+1}; q) = d\Phi_m(k_1, \dots, K_I, K_K, \dots, k_{m+1}; q) \cdot d\Phi_{X_{ijk}}(k_i, k_j, k_k; K_I + K_K). \quad (2.2)$$

For massive radiators, the phase space $d\Phi_m$ is the d -dimensional ($d = 4 - 2\epsilon$) phase space for m outgoing particles with momenta p_1, \dots, p_m , masses m_1, \dots, m_m with total four-momentum q^μ . It reads,

$$d\Phi_m(k_1, \dots, k_m; q) = \frac{d^{d-1}k_1}{2E_1(2\pi)^{d-1}} \cdots \frac{d^{d-1}k_m}{2E_m(2\pi)^{d-1}} (2\pi)^d \delta^d(q - k_1 - \dots - k_m), \quad (2.3)$$

with

$$E_i = \sqrt{|\vec{p}_i|^2 + m_i^2}. \quad (2.4)$$

S_m is a symmetry factor for identical partons in the final state. $d\Phi_{X_{ijk}}$ is the NLO antenna phase space, it is proportional to the three-particle phase space relevant to a $1 \rightarrow 3$ decay. This can be seen by using $m = 2$ in the above formula (2.2) and exploiting the fact that the two-particle phase space is a constant. The massive NLO antenna phase space will be derived in section 5.

The tree-level antenna function X_{ijk}^0 describes all the situations where parton j is unresolved. Those are obtained by normalising the corresponding colour-ordered three-parton tree-level squared matrix elements to the squared matrix element for the basic

two-parton process, omitting all couplings and colour factors. Indeed, the normalisation for the antennae is chosen such that in the unresolved limits those yield exactly the well-known collinear splitting functions and soft eikonal factors. As such the massless and massive antenna functions have mass dimension -2 and are defined by [8],

$$X_{ijk}^0 = S_{ijk,IK} \frac{|\mathcal{M}_{ijk}^0|^2}{|\mathcal{M}_{IK}^0|^2}, \tag{2.5}$$

where S denotes the symmetry factor associated to the antenna, which accounts both for potential identical particle symmetries and for the presence of more than one antenna in the basic two-parton process. The massive antenna functions will be given below in section 3 while their unresolved limits will be presented in section 4.

3 Massive NLO antenna functions

The massive final-final antenna functions can be derived from physical matrix elements of the same processes as in the massless case [8, 10, 11] but considering the final state radiators to be massive. More precisely, the quark-antiquark antenna function which we denote by $A_{Qg\bar{Q}}^0$ (to be distinguished from $A_{qg\bar{q}}^0$) can be derived from $\gamma^* \rightarrow Q\bar{Q} + g$ with Q being a massive quark of mass m_Q , g is a massless gluon. We find three types of massive quark-gluon antennae involving either one or two massive partons. Those can be derived from the following processes: ($\tilde{\chi} \rightarrow \tilde{g} + 2$ partons). For these final state partons being two gluons, we derive D_{Qgg}^0 while for these partons being a quark-antiquark pair, we derive $E_{Qq'\bar{q}'}^0$ and $E_{qQ'\bar{Q}'}^0$. Lastly, we can also define a massive gluon-gluon antenna through the process $H \rightarrow gQ\bar{Q}$, with one of the gluons in the final state emitting a massive $Q\bar{Q}$ pair. All new massive antenna functions are given below with m_Q denoting the mass of the massive quark Q , g is a massless gluon and the invariant s_{ij} is chosen to be $s_{ij} = 2p_i \cdot p_j$ making the mass dependence in the formulae below explicit. E_{cm} is always the rest energy of the decaying particle. All antennae listed below are known exactly at all orders in ϵ . The $O(\epsilon)$ parts have been omitted for conciseness only, those are however already needed at this next-to-leading order level.

The quark-antiquark massive antenna function $A_{Qg\bar{Q}}^0$ reads,

$$\begin{aligned} A_{Qg\bar{Q}}^0(1, 3, 2) = & \frac{1}{4(E_{cm}^2 + 2m_Q^2)} 4 \left(\frac{2s_{12}^2}{s_{13}s_{23}} + \frac{2s_{12}}{s_{13}} + \frac{2s_{12}}{s_{23}} + \frac{s_{23}}{s_{13}} + \frac{s_{13}}{s_{23}} \right. \\ & + m_Q^2 \left(\frac{8s_{12}}{s_{13}s_{23}} - \frac{2s_{12}}{s_{13}^2} - \frac{2s_{12}}{s_{23}^2} - \frac{2s_{23}}{s_{13}^2} - \frac{2}{s_{13}} - \frac{2}{s_{23}} - \frac{2s_{13}}{s_{23}^2} \right) \\ & \left. + m_Q^4 \left(-\frac{8}{s_{23}^2} - \frac{8}{s_{13}^2} \right) \right) + O(\epsilon). \end{aligned} \tag{3.1}$$

This function has been normalised to the two-particle matrix element relevant for $\gamma^* \rightarrow Q\bar{Q}$, whose matrix element squared (omitting couplings) is given by

$$A_{Q\bar{Q}}^0(1, 2) = 4 [(1 - \epsilon) E_{cm}^2 + 2m_Q^2]. \tag{3.2}$$

The quark-gluon massive antennae with either two gluons or a massless quark-antiquark pair in the final state are normalised by the two-particle matrix element squared relevant for $\tilde{\chi} \rightarrow \tilde{g}g$, with the gluino \tilde{g} being massive with mass m_Q , the gluon g being massless. This two-particle matrix element squared omitting couplings reads:

$$X_{Qg}^0(1, 2) = 4(1 - \epsilon) (E_{cm}^2 - m_Q^2)^2, \quad (3.3)$$

where X can stand for E or D here. The quark-gluon massive antenna D_{Qgg}^0 with two gluons and a massive radiator Q in the final state reads,

$$\begin{aligned} D_{Qgg}^0(1, 3, 4) = & \frac{1}{4(E_{cm}^2 - m_Q^2)^2} 4 \left(\left(9s_{13} + 9s_{14} + \frac{4s_{13}^2}{s_{14}} + \frac{4s_{14}^2}{s_{13}} + \frac{4s_{13}^2}{s_{34}} + \frac{2s_{13}^3}{s_{14}s_{34}} \right. \right. \\ & \left. \left. + \frac{6s_{13}s_{14}}{s_{34}} + \frac{4s_{14}^2}{s_{34}} + \frac{2s_{14}^3}{s_{13}s_{34}} + 6s_{34} + \frac{3s_{13}s_{34}}{s_{14}} + \frac{3s_{14}s_{34}}{s_{13}} + \frac{s_{34}^2}{s_{13}} + \frac{s_{34}^2}{s_{14}} \right) \\ & - m_Q^2 \left(6 + \frac{2s_{13}^2}{s_{14}^2} + \frac{4s_{13}}{s_{14}} + \frac{4s_{14}}{s_{13}} + \frac{2s_{14}^2}{s_{13}^2} + \frac{6s_{34}}{s_{13}} + \frac{4s_{13}s_{34}}{s_{14}^2} \right. \\ & \left. + \frac{6s_{34}}{s_{14}} + \frac{4s_{14}s_{34}}{s_{13}^2} + \frac{2s_{34}^2}{s_{13}^2} + \frac{2s_{34}^2}{s_{14}^2} + \frac{2s_{34}^2}{s_{13}s_{14}} \right) \\ & \left. + 2E_{cm}m_Q - E_{cm}m_Q^3 \frac{2s_{34}}{s_{13}s_{14}} + m_Q^4 \frac{2s_{34}}{s_{13}s_{14}} \right) + O(\epsilon). \end{aligned} \quad (3.4)$$

The quark-gluon massive antenna $E_{Qq'\bar{q}'}^0$ with a pair of massless quarks and a massive radiator quark Q in the final state reads,

$$E_{Qq'\bar{q}'}^0(1, 3, 4) = \frac{1}{4(E_{cm}^2 - m_Q^2)^2} 4 \left(s_{13} + s_{14} + \frac{s_{13}^2}{s_{34}} + \frac{s_{14}^2}{s_{34}} - 2E_{cm}m_Q \right) + O(\epsilon). \quad (3.5)$$

In the case where the gluino is massless and the gluon radiates a massive quark-antiquark pair, the two-particle matrix element which serves to normalise the quark-gluon massive antenna $E_{qQ'\bar{Q}'}^0$ is different. It reads:

$$E_{qg}(1, 2) = 4E_{cm}^4 (1 - \epsilon). \quad (3.6)$$

The quark-gluon massive antenna $E_{qQ'\bar{Q}'}^0$ with a pair of massive quarks and a massless quark-radiator in the final state reads:

$$\begin{aligned} E_{qQ'\bar{Q}'}^0(1, 3, 4) = & \frac{1}{4E_{cm}^4} \frac{4}{(s_{34} + 2m_Q^2)^2} (s_{13}^2s_{34} + s_{14}^2s_{34} + s_{13}s_{34}^2 + s_{14}s_{34}^2 \\ & + m_Q^2(4s_{13}^2 + 4s_{13}s_{14} + 4s_{14}^2 + 6s_{13}s_{34} + 6s_{14}s_{34}) \\ & + 8m_Q^4(s_{13} + s_{14})) + O(\epsilon). \end{aligned} \quad (3.7)$$

Finally, the gluon-gluon massive antenna where the final state gluon emits a massive quark-antiquark pair reads:

$$G_{gQ'\bar{Q}'}^0(1, 3, 4) = \frac{1}{4E_{cm}^4} \frac{4}{(s_{34} + 2m_Q^2)^2} (s_{13}^2s_{34} + s_{14}^2s_{34} + 4m_Q^2(s_{13}^2 + s_{14}^2 + s_{13}s_{14})) + O(\epsilon). \quad (3.8)$$

This antenna function has been normalised to the two-particle matrix element relevant for $H \rightarrow gg$ whose matrix element squared is given by,

$$G_{gg}(1, 2) = 4E_{cm}^4 (1 - \epsilon). \quad (3.9)$$

4 Singular limits

The antenna functions listed above encapsulate all single unresolved limits of tree-level QCD matrix elements involving massive final states. The factorisation properties of tree-level QCD squared matrix elements are well-known [15] for massless and for massive final state partons. In the antenna picture, for massless radiators, the unresolved and massless parton emitted between those can be either collinear or soft. In these corresponding unresolved limits, defined as in [8] the $(m + 1)$ -parton matrix element squared factorises into a reduced m -parton matrix element and the well known soft eikonal factor when a gluon is soft and one of the three different Altarelli-Parisi splitting functions [25] when two of the three massless partons are collinear. When the radiators which emit a massless and unresolved parton between them are massive instead, the limiting behaviour of the matrix elements has to be changed and the unresolved factors have to be modified accordingly. In this section we first recall the single unresolved massless factors and present their extension needed to deal with massive radiators. Finally, we list all non-vanishing limits of the three-parton massive antennae defined in section 3.

4.1 Single unresolved massless factors

When a soft gluon (j) is emitted between two hard and massless partons (i and k), the eikonal factor S_{ijk} factorises off the squared matrix element,

$$S_{ijk} \equiv \frac{2s_{ik}}{s_{ij}s_{jk}}. \quad (4.1)$$

When two massless partons become collinear, we have different splitting functions corresponding to various final state configurations: a massless quark splits into a quark and a gluon ($P_{qg \rightarrow Q}$), a gluon splits into a quark-antiquark pair ($P_{q\bar{q} \rightarrow G}$) or a gluon splits into two gluons ($P_{gg \rightarrow G}$). These are given by:

$$\begin{aligned} P_{qg \rightarrow Q}(z) &= \left(\frac{1 + (1 - z)^2 - \epsilon z^2}{z} \right), \\ P_{q\bar{q} \rightarrow G}(z) &= \left(\frac{z^2 + (1 - z)^2 - \epsilon}{1 - \epsilon} \right), \\ P_{gg \rightarrow G}(z) &= 2 \left(\frac{z}{1 - z} + \frac{1 - z}{z} + z(1 - z) \right). \end{aligned} \quad (4.2)$$

In these equations, z is the momentum fraction carried by the unresolved parton and the label q present in these splitting functions can stand for a quark or an antiquark: $P_{qg \rightarrow Q} = P_{\bar{q}g \rightarrow \bar{Q}}$ by charge conjugation. Q or G appearing in the collinear splitting functions denotes the parent particle of the two collinear partons i and j .

4.2 Single unresolved massive factors

When massive partons are present in the final state, those can be soft but they cannot be strictly collinear, the mass is regularizing the collinear divergence. The relation between the massive matrix element squared and the splitting functions needs to be extended from

massless to massive. Similar factorisation formulae as in the massless case will hold provided the collinear limit is generalized [14] to the *quasi-collinear* limit.

The limit when a massive parton P of momentum p^μ decays quasi-collinearly into two massive partons j and k is defined by,

$$p_j^\mu \rightarrow z p^\mu, \quad p_k^\mu \rightarrow (1-z) p^\mu, \quad (4.3)$$

$$p^2 = m_{(jk)}^2. \quad (4.4)$$

with the constraints,

$$p_j \cdot p_k, m_j, m_k, m_{jk} \rightarrow 0 \quad (4.5)$$

at fixed ratios,

$$\frac{m_j^2}{p_j \cdot p_k}, \frac{m_k^2}{p_j \cdot p_k}, \frac{m_{jk}^2}{p_j \cdot p_k}. \quad (4.6)$$

The key difference between the massless collinear limit and the quasi-collinear limit is given by the constraint that the on-shell masses squared have to be kept of the same order as the invariant mass $(p_j + p_k)^2$, with the latter becoming small. In this corresponding quasi-collinear limits, the $(m+1)$ -parton matrix element squared factorises into a reduced m -parton matrix element and unresolved massive factors.

More precisely, the single unresolved massive factors in which the real matrix elements factorise in the soft and quasi-collinear limits are generalizations of the massless soft and collinear Altarelli-Parisi splitting functions defined above. The massive splitting functions denoted by $P_{ij \rightarrow (ij)}(z, \mu_{ij}^2)$ for parton (ij) splitting into partons i and j in d -dimensions ($d = 4 - 2\epsilon$) will be given below. In these expressions, z is the momentum fraction carried by the unresolved parton becoming quasi-collinear to the massive parton. All the mass dependence can be parameterized by $\mu_{jk} = (m_j^2 + m_k^2)/[(p_j + p_k)^2 - m_{(jk)}^2]$. The number of gluon polarizations is chosen within the dimensional regularisation procedure as $d - 2$. Those generalized massive splitting functions were also given in the appendix of [16] and read,

$$P_{qg \rightarrow Q}(z, \mu_{qg}^2) = \left(\frac{1 + (1-z)^2 - \epsilon z^2}{z} - 2\mu_{Qg}^2 \right)$$

where μ_{qg}^2 is defined by $\mu_{qg}^2 = \frac{m_Q^2}{s_{qg}}$,

$$P_{q\bar{q} \rightarrow G}(z, \mu_{q\bar{q}}^2) = \left(\frac{z^2 + (1-z)^2 - \epsilon - \mu_{q\bar{q}}^2}{1-\epsilon} \right)$$

where $\mu_{q\bar{q}}^2$ is defined by $\mu_{q\bar{q}}^2 = \frac{2m_Q^2}{[s_{q\bar{q}} + 2m_Q^2]}$.

The gluon-gluon splitting function $P_{gg \rightarrow G}(z)$ is left unchanged. The generalized soft eikonal factor $S_{ijk}(m_i, m_k)$ depends on the invariants $s_{lm} = 2p_l \cdot p_m$ built with the partons i, j and k but also on the masses of partons i and k , m_i and m_k respectively. It is given by,

$$S_{ijk}(m_i, m_k) = \frac{2s_{ik}}{s_{ij}s_{jk}} - \frac{2m_i^2}{s_{ij}^2} - \frac{2m_k^2}{s_{jk}^2}. \quad (4.7)$$

4.3 Singular limits of the massive NLO antenna functions

We list here the non-vanishing soft, collinear and quasi-collinear limits of the massive antenna functions given in section 3. The singular limits of $A_{Qg\bar{Q}}^0$ are

$$\begin{aligned} A_{Qg\bar{Q}}^0(1, 3, 2) &\xrightarrow{3_g \rightarrow 0} \mathcal{S}_{132}(m_Q, m_Q), \\ A_{Qg\bar{Q}}^0(1, 3, 2) &\xrightarrow{3_g \parallel 1_Q} \frac{1}{s_{13}} P_{qg \rightarrow Q}(z, \mu_{qg}^2), \\ A_{Qg\bar{Q}}^0(1, 3, 2) &\xrightarrow{3_g \parallel 2_{\bar{Q}}} \frac{1}{s_{23}} P_{qg \rightarrow Q}(z, \mu_{qg}^2). \end{aligned} \tag{4.8}$$

For D_{Qgg}^0 we have

$$\begin{aligned} D_{Qgg}^0(1, 3, 4) &\xrightarrow{3_g \rightarrow 0} \mathcal{S}_{134}(m_Q, 0), \\ D_{Qgg}^0(1, 3, 4) &\xrightarrow{3_g \parallel 1_Q} \frac{1}{s_{13}} P_{qg \rightarrow Q}(z, \mu_{qg}^2), \\ D_{Qgg}^0(1, 3, 4) &\xrightarrow{4_g \parallel 1_Q} \frac{1}{s_{14}} P_{qg \rightarrow Q}(z, \mu_{qg}^2), \\ D_{Qgg}^0(1, 3, 4) &\xrightarrow{3_g \parallel 4_g} \frac{1}{s_{34}} P_{gg \rightarrow G}(z). \end{aligned} \tag{4.9}$$

The only non-vanishing singular limit of $E_{Qq'\bar{q}'}^0$ is the collinear massless limit of the massless quark-antiquark pair,

$$E_{Qq'\bar{q}'}^0(1, 3, 4) \xrightarrow{3_{q'} \parallel 4_{\bar{q}'}} \frac{1}{s_{34}} P_{q\bar{q} \rightarrow G}(z), \tag{4.10}$$

while the only non-vanishing limit of $E_{qQ'\bar{Q}'}^0$ is,

$$E_{qQ'\bar{Q}'}^0(1, 3, 4) \xrightarrow{3_{Q'} \parallel 4_{\bar{Q}'}} \frac{1}{s_{34} + 2m_Q^2} P_{q\bar{q} \rightarrow G}(z, \mu_{q\bar{q}}^2). \tag{4.11}$$

Finally $G_{gQ'\bar{Q}'}^0$ does also have a quasi-collinear quark-antiquark limit given by

$$G_{gQ'\bar{Q}'}^0(1, 3, 4) \xrightarrow{3_{Q'} \parallel 4_{\bar{Q}'}} \frac{1}{s_{34} + 2m_Q^2} P_{q\bar{q} \rightarrow G}(z, \mu_{q\bar{q}}^2). \tag{4.12}$$

5 Integrated massive NLO antenna functions

In this section we present the results for the integration over the appropriate antenna phase space of the massive final-final three-parton antenna functions defined in section 3.

As in the massless case, the phase space associated to the subtraction term given in eq. (2.2) can be factorised. In a first step, we present the phase space factorisation of a three-particle phase space with three massive particles into a two-particle phase space and an antenna phase space. The latter is proportional to a three particle massive phase space. We need to consider two cases: three particle final state with one massless parton and either two massive partons of equal mass, or one massive parton and two massless partons.

5.1 Phase space factorisation

In the rest frame of a decaying photon of momentum q , the d -dimensional two-particle phase space $d\Phi_2(p_I, p_K; q)$ given in eq. (2.3) for two outgoing particles I and K with momenta p_I and p_K and masses m_I and m_K reads:

$$d\Phi_2((p_I, p_K; q) = (2\pi)^{2-d} \left(\frac{1}{2}\right)^{d-1} (E_{cm}^2)^{1-d/2} ((E_{cm}^2 - m_I^2 - m_K^2)^2 - 4m_I^2 m_K^2)^{\frac{d-3}{2}} d\Omega_{d-1}, \quad (5.1)$$

with

$$\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}, \quad (5.2)$$

where we have used

$$\int \frac{d^{d-1}p_I}{2E_I} = \frac{1}{2} \int dE_I |\vec{p}_I|^{d-3} d\Omega_{d-1}. \quad (5.3)$$

In this last equation, Ω_{d-1} parameterizes the solid angle of the $d-1$ components of \vec{p}_I in $d-1$ dimensions. The three particle phase space $d\Phi_3(p_i, p_j, p_k; q)$ may be written as the product of a two particle phase space $d\Phi_2(p_I, p_K; q)$ defined above and an antenna phase space $d\Phi_{X_{ijk}}$ which depends on the momenta p_i, p_j and p_k and on the masses m_i, m_j and m_k .

Using spherical coordinates in d dimensions, we have

$$\int \frac{d^{d-1}p_i}{2E_i} \frac{d^{d-1}p_j}{2E_j} = \int \frac{1}{4} [|\vec{p}_i||\vec{p}_j| \sin \theta]^{d-3} dE_i dE_j d\theta d\Omega_{d-2} d\Omega_{d-1} \quad (5.4)$$

with θ being the angle between the vectors \vec{p}_i and \vec{p}_j . $\Omega_{d-1}, \Omega_{d-2}$ parameterize the solid angles of the $d-1$ and $d-2$ components of the vectors \vec{p}_i and \vec{p}_j respectively, in $d-1$ dimensions. We find,

$$\int d\Phi_3(p_i, p_j, p_k; q) = \int d\Phi_2(p_I, p_K; q) \times d\Phi_{X_{ijk}} \quad (5.5)$$

with the antenna phase space $d\Phi_{X_{ijk}}$ given by,

$$\begin{aligned} \int d\Phi_{X_{ijk}}(s_{ij}, s_{jk}, s_{ik}) &= (2\pi)^{1-d} \frac{2\pi^{d/2-1}}{\Gamma(\frac{d}{2}-1)} \frac{1}{4} ((E_{cm}^2 - m_I^2 - m_K^2)^2 - 4m_I^2 m_K^2)^{\frac{3-d}{2}} \\ &\int ds_{ij} ds_{jk} ds_{ik} \delta(E_{cm}^2 - m_i^2 - m_j^2 - m_k^2 - s_{ij} - s_{jk} - s_{ik}) \\ &[4 \Delta_3(p_i, p_j, p_k)]^{\frac{d-4}{2}} \theta(\Delta_3(p_i, p_j, p_k)). \end{aligned} \quad (5.6)$$

The masses m_I and m_K appearing in this equation are combinations of the masses m_i, m_j and m_k and the integral over Ω_{d-2} has been performed. The function $\Delta_3(p_i, p_j, p_k)$ is the Gram determinant for massive particles of momenta p_i, p_j, p_k given in terms of invariants $s_{ij} = 2p_i \cdot p_j$ and masses m_i, m_j, m_k by,

$$\Delta_3(p_i, p_j, p_k) = \frac{1}{4} (s_{ij} s_{ik} s_{jk} - m_i^2 s_{jk}^2 - m_k^2 s_{ij}^2 - m_j^2 s_{ik}^2 + 4m_i^2 m_j^2 m_k^2) \quad (5.7)$$

As in the massless case, the factorisation of the phase space is exact. The expression for the antenna phase space $d\Phi_{X_{ijk}}$ given in eq. (5.6) can also be found in [14, 17], where it

was called a dipole phase space. At NLO, both massless and massive dipole and antenna phase spaces are the same. The NLO massive antenna functions differ however from the NLO massive dipoles functions given [14] as did the NLO massless antenna [8] from the massless dipole functions [6].

In the following, we will only consider two kinematical configurations of the final state particles. For the case where one of mass vanishes ($m_j = 0$) and the two other masses are equal ($m_i = m_k = m_Q$), i.e $m_I = m_K = m_i$ in eq. (5.6), we use the following parameterization,

$$\begin{aligned}
 d\Phi_{X_{ijk}}^{(m,0,m)} &= \frac{(4\pi)^{\epsilon-2}}{\Gamma(1-\epsilon)} (E_{cm}^2)^{1-\epsilon} (r_0)^{2-2\epsilon} 2^{2\epsilon-1} \\
 &\quad \int_0^1 dr r^{1-2\epsilon} (1-r)^{-\epsilon+\frac{1}{2}} (1-r_0r)^{-\frac{1}{2}} \int_{-1}^1 ds (1-s^2)^{-\epsilon}, \\
 r_0 &= 1 - \frac{4m_Q^2}{E_{cm}^2}, \\
 s_{ij} &= \frac{1}{2} E_{cm}^2 r_0 r \left(1 - s \sqrt{\frac{r_0(1-r)}{1-r_0r}} \right), \\
 s_{jk} &= \frac{1}{2} E_{cm}^2 r_0 r \left(1 + s \sqrt{\frac{r_0(1-r)}{1-r_0r}} \right),
 \end{aligned} \tag{5.8}$$

and for the case where two particles are massless $m_i = m_j = 0$ and the third one is massive, $m_k = m_Q$, in which case $m_I = m_i = 0$, $m_K = m_k$ in eq. (5.6), we use,

$$\begin{aligned}
 d\Phi_{X_{ijk}}^{(0,0,m)} &= \frac{(4\pi)^{\epsilon-2}}{\Gamma(1-\epsilon)} (E_{cm}^2)^{1-\epsilon} (u_0)^{2-2\epsilon} \\
 &\quad \int_0^1 du u^{1-2\epsilon} (1-u)^{1-2\epsilon} (1-u_0u)^{-1+\epsilon} \int_0^1 dv v^{-\epsilon} (1-v)^{-\epsilon}, \\
 u_0 &= 1 - \frac{m_Q^2}{E_{cm}^2}, \\
 s_{ij} &= E_{cm}^2 u_0^2 \frac{u(1-u)v}{(1-u_0u)}, \\
 s_{ik} &= E_{cm}^2 u_0(1-u).
 \end{aligned} \tag{5.9}$$

5.2 Integrals

Following the extension of the integration by parts method [26, 27] in [28, 29], to reduce the number of real phase space integrals, we have expressed all invariants in terms of massive propagators and expressed the three on-shell conditions $p_i^2 = m_i^2$, $i = 1, 2, 3$ as cut propagators. The reduction to master integrals using the Laporta algorithm [30] was done independently, once with an in-house implementation in FORM [31] and once with the mathematica package FIRE [32]. We find five master integrals, four of which can be evaluated in terms of hypergeometric functions for arbitrary ϵ . For the last one, only the expansion up to order ϵ^2 will be given. Differential equation techniques [33, 34] are used to compute this master integral.

The integrated antennae can be separated in two categories corresponding to the phase space parameterizations used. The integrated antennae with two massive particles are obtained by integrating the antennae with two massive final state partons presented in section 3 over the antenna phase space given in eq. (5.8). Those can be expressed in terms of two master integrals $I_i^{(m,0,m)}$ ($i=1,2$) and are given exactly at any order in ϵ by,

$$\begin{aligned}
 \mathcal{A}_{Qg\bar{Q}}^0 &= \frac{1}{E_{cm}^2 \epsilon (1-2\epsilon) r_0 (1-r_0) (3-r_0-2\epsilon)} \\
 &\quad \left((2(1-\epsilon)(-15+8r_0-r_0^2+\epsilon(28-12r_0+4r_0^2)-\epsilon^2(12+4r_0+4r_0^2)+8\epsilon^3 r_0)) I_1^{(m,0,m)} \right. \\
 &\quad \left. + \frac{1}{E_{cm}^2} 24(1-\epsilon)^2 (3-r_0-\epsilon(2+r_0)+2\epsilon^2 r_0) I_2^{(m,0,m)} \right), \\
 \mathcal{E}_{gQ'\bar{Q}'}^0 &= \frac{1}{E_{cm}^2} \left(\frac{2r_0(3+r_0-4\epsilon r_0)}{3(1-r_0)} I_1^{(m,0,m)} - \frac{1}{E_{cm}^2} 4 \frac{(1+r_0-2\epsilon r_0)}{(1-r_0)} I_2^{(m,0,m)} \right), \\
 \mathcal{G}_{gQ'\bar{Q}'}^0 &= \frac{1}{E_{cm}^2} \left(\frac{8(1-\epsilon)r_0^2}{3(1-r_0)} I_1^{(m,0,m)} - \frac{1}{E_{cm}^2} 8(1-\epsilon) \frac{r_0}{(1-r_0)} I_2^{(m,0,m)} \right). \tag{5.10}
 \end{aligned}$$

The two master integrals $I_i^{(m,0,m)}$ are given exactly below. $I_1^{(m,0,m)}$ is the integrated phase space measure and is given by

$$\begin{aligned}
 I_1^{(m,0,m)} &= \int d\Phi_{X_{ijk}}^{(m,0,m)} \\
 &= (E_{cm}^2)^{1-\epsilon} r_0^{2-2\epsilon} 2^{-2\epsilon} \pi^{-2+\epsilon} \frac{\Gamma(2-2\epsilon)\Gamma(3-3\epsilon)}{\Gamma(6-6\epsilon)} {}_2F_1 \left(\frac{1}{2}, 2-2\epsilon, \frac{7}{2}-3\epsilon; r_0 \right), \tag{5.11}
 \end{aligned}$$

and

$$\begin{aligned}
 I_2^{(m,0,m)} &= \int d\Phi_{X_{ijk}}^{(m,0,m)}(s_{ij}) \\
 &= (E_{cm}^2)^{2-\epsilon} r_0^{3-2\epsilon} 2^{1-2\epsilon} \pi^{-2+\epsilon} \frac{\Gamma(3-2\epsilon)\Gamma(4-3\epsilon)}{\Gamma(8-6\epsilon)} {}_2F_1 \left(\frac{1}{2}, 3-2\epsilon, \frac{9}{2}-3\epsilon; r_0 \right), \tag{5.12}
 \end{aligned}$$

with $p_i^2 = m_i^2$ and $p_j^2 = 0$.

The integrated antennae with one massive particle are obtained by integrating the antenna functions with one massive final state presented in section 3 with the phase space given in eq. (5.9) (with $\mu = \frac{m_Q}{E_{cm}}$). Those can be expressed in term of three master integrals $I_i^{0,0,m}$ ($i=1, \dots, 3$) as follows,

$$\begin{aligned}
 \mathcal{D}_{Qgg}^0 &= \frac{1}{E_{cm}^2} \frac{1}{\epsilon^2} \frac{1}{(1-\epsilon)(1-2\epsilon)\mu^2(1-\mu)^2(1+\mu)^3} (12\mu^4(1+\mu) \\
 &\quad + 2\epsilon(1+\mu)(1-6\mu^2-28\mu^4) - \epsilon^2(7+7\mu-42\mu^2-40\mu^3-111\mu^4-125\mu^5) \\
 &\quad + 2\epsilon^3(5+3\mu-31\mu^2-22\mu^3-44\mu^4-59\mu^5) \\
 &\quad - \epsilon^4(1+\mu)(7-8\mu-38\mu^2+30\mu^3-19\mu^4) \\
 &\quad + 2\epsilon^5(1+\mu)^2(1-3\mu-\mu^2+9\mu^3)) I_1^{(0,0,m)} \\
 &\quad + \frac{1}{E_{cm}^4} \frac{1}{\epsilon^2} \frac{1}{(1-2\epsilon)\mu^2(1-\mu)^2(1+\mu)^3} (12\mu^2(1+\mu) - 2\epsilon(1+\mu)(3+22\mu^2)
 \end{aligned}$$

$$\begin{aligned}
 & + \epsilon^2(15 + 15\mu + 65\mu^2 + 77\mu^3) - \epsilon^3(1 + \mu)(15 - 12\mu + 31\mu^2) \\
 & + 2\epsilon^4(1 + \mu)(3 - 6\mu - 7\mu^2) I_2^{(0,0,m)} \\
 & - E_{cm}^2 \frac{1}{1 - \epsilon} \frac{2\mu}{1 + \mu} (\mu(1 + 2\mu) + \epsilon(1 - \mu^2) - \epsilon^2(1 + \mu)^2) I_3^{(0,0,m)}, \\
 \mathcal{E}_{Qq'\bar{q}'}^0 = & \frac{2}{\epsilon E_{cm}^2 (1 - \mu^2)^2} \left((-2\mu^2 + \epsilon(3\mu^2 - \mu)) I_1^{(0,0,m)} - \frac{2 - 3\epsilon}{E_{cm}^2} I_2^{(0,0,m)} \right). \quad (5.13)
 \end{aligned}$$

The master integral $I_1^{(0,0,m)}$ corresponding to the integrated phase space and the master integral $I_2^{(0,0,m)}$ are given exactly by,

$$\begin{aligned}
 I_1^{(0,0,m)} & = \int d\Phi_{X_{ijk}}^{(0,0,m)} \quad (5.14) \\
 & = (E_{cm}^2)^{1-\epsilon} u_0^{2-2\epsilon} 2^{-4+2\epsilon} \pi^{-2+\epsilon} \frac{\Gamma(2-2\epsilon)\Gamma(1-\epsilon)}{\Gamma(4-4\epsilon)} {}_2F_1(1-\epsilon, 2-2\epsilon, 4-4\epsilon; u_0),
 \end{aligned}$$

$$\begin{aligned}
 I_2^{(0,0,m)} & = \int d\Phi_{X_{ijk}}^{(0,0,m)}(s_{ik}) \quad (5.15) \\
 & = (E_{cm}^2)^{2-\epsilon} u_0^{3-2\epsilon} 2^{-5+2\epsilon} \pi^{-2+\epsilon} \frac{\Gamma(2-2\epsilon)\Gamma(1-\epsilon)}{\Gamma(4-4\epsilon)} {}_2F_1(1-\epsilon, 2-2\epsilon, 5-4\epsilon; u_0),
 \end{aligned}$$

with $p_i^2 = 0, p_k^2 = m_k^2$.

The third master integral $I_3^{(0,0,m)}$ is defined as follows,

$$\begin{aligned}
 I_3^{(0,0,m)} & = \int \frac{1}{s_{ik}s_{jk}} d\Phi_{X_{ijk}}^{(0,0,m)} \quad (5.16) \\
 & = \frac{(E_{cm}^2)^{-1-\epsilon} (4\pi)^{-2+\epsilon} u_0^{-2\epsilon}}{\Gamma(1-\epsilon)} \int_0^1 \int_0^1 dudv \frac{u^{-2\epsilon}(1-u)^{-2\epsilon}(1-u_0u)^\epsilon v^{-\epsilon}(1-v)^{-\epsilon}}{1-u_0u - (1-u)u_0v}.
 \end{aligned}$$

To get the integrated quark-gluon antenna denoted by \mathcal{D}_{Qgg}^0 up to finite terms, only the constant term of the integral $I_3^{(0,0,m)}$ is needed. The evaluation of this integral yields hypergeometric functions of argument u_0 . The expansion around $\epsilon = 0$ of these hypergeometric functions leads to one-dimensional harmonic polylogarithms [35–37], *HPL*-functions, denoted here by H . The constant term in $I_3^{(0,0,m)}$ yields,

$$I_3^{(0,0,m)} \Big|_{\epsilon=0} = \frac{1}{E_{cm}^2} \frac{1}{16\pi^2 u_0} (H(1, 1; u_0) + H(2; u_0)). \quad (5.17)$$

Foreseeing that the results we obtained for the integrated NLO massive antenna functions could be used as an input for a further development of NNLO antenna subtraction involving massive radiators we derived the order ϵ and ϵ^2 of this master integral $I_3^{(0,0,m)}$.

For this purpose, we derived differential equations for I_3 with respect to E_{cm}^2 and u_0 . The equations obtained are homogeneous with respect to E_{cm}^2 and inhomogeneous with respect to u_0 . In the differential equation for I_3 with respect to u_0 the master integrals I_1 and I_2 appear as coefficients of the inhomogeneous part of the equation. By inserting their known results, we could solve the differential equation for I_3 order by order in ϵ . Some

care has to be taken however. These master integrals I_1, I_2 and I_3 are obtained after an integration over the antenna phase space $d\Phi_{X_{ijk}}^{(0,0,m)}$. The differential equation with respect to u_0 however concerns integrals over the full three-parton phase space.

As we saw in eq. (5.5) the latter phase space can be written as the product of the antenna phase space and a two-particle phase space. However, since the two particle phase space depends on u_0 , special care has to be taken solving the differential equation for u_0 . Solving this differential equation will yield the wanted master integral I_3 multiplied by the two-particle phase space with one massive and one massless particle denoted by $\Phi_2(m, 0)$. The master integrals I_1 and I_2 defined above will not appear as such in the differential equation but those appear as multiplied by the two-particle phase space $\Phi_2(m, 0)$.

More precisely, the two-particle phase space with one massive and one massless particle reads,

$$\Phi_2(m, 0) = 2^{-3+2\epsilon} (\pi)^{-1+\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)} (u_0)^{1-2\epsilon} (E_{cm}^2)^{-\epsilon}. \quad (5.18)$$

The master integrals $I_1(u_0)$ and $I_2(u_0)$ appearing in the differential equation for I_3 are obtained as the product of the original master integrals I_1 and I_2 with the two particle phase space measure $\Phi_2(m, 0)$. Those take the form,

$$\begin{aligned} I_1(u_0) &= O_{cc} (E_{cm}^2)^{1-2\epsilon} (u_0)^{3-4\epsilon} {}_2F_1(1-\epsilon, 2-2\epsilon, 4-4\epsilon; u_0) \\ I_2(u_0) &= O_{cc} (E_{cm}^2)^{2-2\epsilon} (u_0)^{4-4\epsilon} {}_2F_1(1-\epsilon, 2-2\epsilon, 5-4\epsilon; u_0) \end{aligned} \quad (5.19)$$

with the overall normalisation factor O_{cc} given by,

$$O_{cc} = 2^{-7+4\epsilon} (\pi)^{-3+2\epsilon} \frac{\Gamma(1-\epsilon)^2}{\Gamma(4-4\epsilon)}. \quad (5.20)$$

With this notation, the master integral I_3 is given up to order ϵ^2 by,

$$I_3 = \frac{1}{\Phi_2(m, 0)} O_{cc} \hat{I} (E_{cm}^2)^{-1-2\epsilon} \quad (5.21)$$

with,

$$\begin{aligned} \hat{I} &= (6H(2; u_0) + 6H(1, 1; u_0) - 2\epsilon(16H(2; u_0) + 6H(3; u_0) + 16H(1, 1; u_0) \\ &\quad - 3H(1, 2; u_0) + 12H(2, 0; u_0) + 3H(2, 1; u_0) + 12H(1, 1, 0; u_0) - 6H(1, 1, 1; u_0)) \\ &\quad + \epsilon^2(16H(2; u_0) + 32H(3; u_0) + 12H(4; u_0) + 16H(1, 1; u_0) - 16H(1, 2; u_0) \\ &\quad - 6H(1, 3; u_0) + 64H(2, 0; u_0) + 16H(2, 1; u_0) + 24H(3, 0; u_0) + 6H(3, 1; u_0) \\ &\quad + 64H(1, 1, 0; u_0) - 32H(1, 1, 1; u_0) + 9H(1, 1, 2; u_0) - 12H(1, 2, 0; u_0) \\ &\quad - 3H(1, 2, 1; u_0) + 48H(2, 0, 0; u_0) + 12H(2, 1, 0; u_0) - 9H(2, 1, 1; u_0) \\ &\quad + 48H(1, 1, 0, 0; u_0) - 24H(1, 1, 1, 0; u_0) + 9H(1, 1, 1, 1; u_0)). \end{aligned} \quad (5.22)$$

6 Check of $\mathcal{A}_{Qg\bar{Q}}^0$

Our new result for the integrated NLO massive quark-antiquark antenna $\mathcal{A}_{Qg\bar{Q}}^0$ given in eq. (5.10) can be tested with expressions known in the literature. The integrated antenna

function $\mathcal{A}_{Qg\bar{Q}}^0$ can be regarded as the order α_s part of the real radiation correction to the decay rate of a virtual photon into a massive quark-antiquark pair, $\gamma^* \rightarrow Q\bar{Q}$. By adding the real corrections obtained with $\mathcal{A}_{Qg\bar{Q}}^0$ to the order α_s part of the virtual corrections for this decay rate, known in the literature, [39–41] we are able to reproduce the known result for the total hadronic decay width at order α_s [41, 42]. The details of the comparison are given below. All formulas are restricted to vector coupling only and one heavy flavour f of quarks Q .

The decay width of a virtual photon (γ^*) into a quark-antiquark pair is given by,

$$\Gamma_{\gamma^*}^{\text{had}} = \frac{1}{2} \alpha E_{cm} N_C Q_f^2 r_{\text{NS}}^V(f). \quad (6.1)$$

with the colour factor C_F given by $C_F = \frac{N_c^2 - 1}{2N_c}$, N_c the number of colours and,

$$r_{\text{NS}}^V(f) = v \frac{3 - v^2}{2} \left[1 + C_F \frac{\alpha_s}{\pi} K_V \right] \quad (6.2)$$

$$K_V = \frac{1}{v} \left(A(v) + \frac{33}{24} + \frac{22}{24} v^2 - \frac{7}{24} v^4 \ln \frac{1+v}{1-v} + \frac{5}{4} v - \frac{3}{4} v^3 \right) \quad (6.3)$$

$$A(v) = (1 + v^2) \left[\text{Li}_2 \left(\left[\frac{1-v}{1+v} \right]^2 \right) + 2 \text{Li}_2 \left(\frac{1-v}{1+v} \right) + \ln \frac{1+v}{1-v} \ln \frac{(1+v)^3}{8v^2} \right] + 3v \ln \frac{1-v^2}{4v} - v \ln v. \quad (6.4)$$

In this equation, $r_0 = 1 - \frac{4m^2}{E_{cm}^2}$ is used as the dimensionless variable and $v = \sqrt{r_0}$. This result is obtained from the decay width of Z boson into a quark-antiquark pair [42] by appropriate coupling replacements. It can be reproduced by computing the vertex correction $\Gamma_{\gamma^*, \text{vertex}}^{\text{had}}$ and the real radiation correction $\Gamma_{\gamma^*, \text{real}}^{\text{had}}$ separately. The corresponding formulae will be given below.

The vertex correction $\Gamma_{\gamma^*, \text{vertex}}^{\text{had}}$ is given in [39, 41] in the form of the QCD vertex amplitude $V_{c_1 c_2}^\mu$ corresponding to the decay of a virtual photon with momentum $p_1 + p_2$ into a quark and an antiquark with colours c_1 and c_2 (and $\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]$)

$$V_{c_1 c_2}^\mu(p_1, p_2) = \bar{u}_{c_1}(p_1) \Gamma_{c_1 c_2}^\mu(p_1, p_2) v_{c_2}(p_2), \quad (6.5)$$

$$\Gamma_{c_1 c_2}^\mu(p_1, p_2) = -i\sqrt{4\pi\alpha} Q_f \delta_{c_1 c_2} \left(F_1 \left(\frac{E_{cm}^2}{m^2} \right) \gamma^\mu + \frac{1}{2m} F_2 \left(\frac{E_{cm}^2}{m^2} \right) i\sigma^{\mu\nu} (p_1 + p_2)_\nu \right), \quad (6.6)$$

where Q_f is the quark charge in units of charge $e = \sqrt{4\pi\alpha}$ for one flavour f . The hadronic vertex correction $\Gamma_{\gamma^*, \text{vertex}}^{\text{had}}$ then reads,

$$\Gamma_{\gamma^*, \text{vertex}}^{\text{had}} = \frac{1}{2E_{cm}} \frac{1}{2} \sum_{\text{spins, colours}} V_{c_1 c_2}^\mu V_{\mu c_3 c_4} \Phi_2. \quad (6.7)$$

Φ_2 is the integrated two-particle phase space for two massive particles given by,

$$\Phi_2(m, m) = 2^{-3+2\epsilon} (\pi)^{-1+\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)} (r_0)^{1/2-\epsilon} (E_{cm}^2)^{-\epsilon} \quad (6.8)$$

and the amplitude squared is,

$$\sum_{spins, colours} V_{c_1 c_2}^\mu V_{\mu c_3 c_4} = N_C (4\pi\alpha Q_f^2) E_{cm}^2 \left(|F_1|^2 (2(3 - r_0 - 2\epsilon)) \right. \tag{6.9}$$

$$\left. + |F_2|^2 \left(\frac{2}{1 - r_0} (3 - 2r_0 - 2\epsilon(1 - r_0)) + \Re(F_1 F_2^*) (4(3 - 2\epsilon)) \right) \right).$$

The form factors F_1 and F_2 are given up to order α_s and ϵ^0 by,

$$F_1 = 1 + C(\epsilon) \frac{\alpha_s}{2\pi} \mathcal{F}_{1,R}^{1l} \tag{6.10}$$

$$F_2 = C(\epsilon) \frac{\alpha_s}{2\pi} \mathcal{F}_{2,R}^{1l} \tag{6.11}$$

with the normalisation factor $C(\epsilon)$ given by,

$$C(\epsilon) = (4\pi)^\epsilon \Gamma(1 + \epsilon) \left(\frac{\mu^2}{m_Q^2} \right)^\epsilon \tag{6.12}$$

and,

$$\mathcal{F}_{1,R}^{1l} = C_F \left(\frac{1}{\epsilon} \left[-1 + \left(1 - \frac{1}{1-y} - \frac{1}{1+y} \right) H(0; y) \right] - 2 - \left(\frac{1}{2} - \frac{1}{1-y} \right) H(0; y) \right. \tag{6.13}$$

$$\left. - \left(1 - \frac{1}{1-y} - \frac{1}{1+y} \right) [4\zeta(2) - 2H(0; y) - H(0, 0; y) - 2H(1, 0; y)] \right),$$

$$\mathcal{F}_{2,R}^{2l} = C_F \left(\left[\frac{1}{1-y} - \frac{1}{1+y} \right] H(0; y) \right).$$

We quote the one-loop renormalised vertex corrections for $E_{cm}^2 > 4m^2$ given in [39] up to order α_s and ϵ^0 respectively. (Imaginary parts are omitted because they do not contribute to $O(\alpha_s)$). The dimensionless variable y is defined by

$$y = \frac{1 - \sqrt{r_0}}{1 + \sqrt{r_0}}.$$

The real emission contribution to the decay width $\Gamma_{\gamma^*, real}^{had}$ is given by

$$\Gamma_{\gamma^*, real}^{had} = \frac{1}{2E_{cm}} \left(\frac{1}{2} \sum_{spins, colours} |\mathcal{M}_{\gamma^* \rightarrow q\bar{q}}^{tree}|^2 \right) \Phi_2(4\pi\alpha_s \mu^{2\epsilon}) \left(2C_F \mathcal{A}_{Qg\bar{Q}}^0 \right), \tag{6.14}$$

while the averaged leading order matrix element squared proportional to $|\mathcal{M}_{\gamma^* \rightarrow q\bar{q}}^{tree}|^2$ reads,

$$\sum_{spins, colours} |\mathcal{M}_{\gamma^* \rightarrow q\bar{q}}^{tree}|^2 = (4\pi\alpha Q_f^2) N_C E_{cm}^2 2(3 - r_0 - 2\epsilon). \tag{6.15}$$

By expanding in ϵ our result for $\mathcal{A}_{Qg\bar{Q}}^0$ given in terms of hypergeometric functions we could show that,

$$\Gamma_{\gamma^*, real}^{had} + \Gamma_{\gamma^*, vertex}^{had} = \Gamma_{\gamma^*}^{had} \tag{6.16}$$

giving us a strong check of our integrated massive final-final $Q\bar{Q}$ antenna $\mathcal{A}_{Qg\bar{Q}}^0$.

7 Conclusions and outlook

We have generalized the antenna subtraction method, originally developed for massless final states to real radiation off massive final state partons at the next-to-leading order level. The main building blocks of the antenna subtraction method are the antenna functions. Those encapsulate all unresolved radiation emitted between two colour-ordered hard radiators. As such those functions account for all unresolved radiation of the corresponding QCD matrix elements. In section 3, we have presented the final-final antenna functions for the following cases: the presence of two massive final state radiators of equal masses or the presence of one massive and one massless radiator. Section 4 contained a list of all non-vanishing unresolved limits of the antennae functions presented in section 3. Explicit phase space factorisation and parameterization formulae were presented in section 5 where all massive final-final antennae functions were integrated over their corresponding phase space measure.

An important extension of subtraction methods is the combination with parton shower algorithms [20–23], thus allowing for a full partonic event generation to NLO accuracy. So far, our massless final-final antenna functions are part of the parton shower VINCIA [18]. With the formulation of the antenna subtraction method for final state radiation off massive fermions presented here, it will become possible to construct antenna-based parton showers involving massive final state particles.

Acknowledgments

We wish to thank Thomas Gehrmann for useful discussions. This research was supported by the Swiss National Science Foundation (SNF) under contract PP002-118864 which is hereby acknowledged.

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